Quenched Lyapunov exponent for the parabolic Anderson model in a dynamic random environment

J. Gärtner, F. den Hollander and G. Maillard

Abstract We continue our study of the parabolic Anderson equation $\partial u/\partial t = \kappa\Delta u + \gamma\xi u$ for the space-time field $u\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$, where $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian, $\gamma \in (0,\infty)$ is the coupling constant, and $\xi\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ is a space-time random environment that drives the equation. The solution of this equation describes the evolution of a "reactant" u under the influence of a "catalyst" ξ, both living on $\mathbb{Z}^d$.

In earlier work we considered three choices for ξ: independent simple random walks, the symmetric exclusion process, and the symmetric voter model, all in equilibrium at a given density. We analyzed the *annealed* Lyapunov exponents, i.e., the exponential growth rates of the successive moments of u w.r.t. ξ, and showed that these exponents display an interesting dependence on the diffusion constant κ, with qualitatively different behavior in different dimensions d. In the present paper we focus on the *quenched* Lyapunov exponent, i.e., the exponential growth rate of u conditional on ξ.

We first prove existence and derive qualitative properties of the quenched Lyapunov exponent for a general ξ that is stationary and ergodic under translations in space and time and satisfies certain noisiness conditions. After that we focus on the three particular choices for ξ mentioned above and derive some further properties. We close by formulating open problems.

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1 Introduction

Section 1.1 defines the parabolic Anderson model, Section 1.2 introduces the quenched Lyapunov exponent, Section 1.3 summarizes what is known in the literature, Section 1.4 contains our main results, while Section 1.5 provides a discussion of these results and lists open problems.

1.1 Parabolic Anderson model

The parabolic Anderson model (PAM) is the partial differential equation

$$\frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + [\gamma \xi(x,t) - \delta] u(x,t), \qquad x \in \mathbb{Z}^d,\ t \geq 0. \tag{1}$$

Here, the u-field is $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian acting on u as

$$\Delta u(x,t) = \sum_{\substack{y \in \mathbb{Z}^d \\ \|y-x\|=1}} [u(y,t) - u(x,t)] \tag{2}$$

($\|\cdot\|$ is the Euclidian norm), $\gamma \in [0,\infty)$ is the coupling constant, $\delta \in [0,\infty)$ is the killing constant, while

$$\xi = (\xi_t)_{t \geq 0} \text{ with } \xi_t = \{\xi(x,t)\colon\ x \in \mathbb{Z}^d\} \tag{3}$$

is an $\mathbb{R}$-valued random field that evolves with time and that drives the equation. The ξ-field provides a dynamic random environment defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. As initial condition for (1) we take

$$u(x,0) = \delta_0(x), \qquad x \in \mathbb{Z}^d. \tag{4}$$

One interpretation of (1) and (4) comes from population dynamics. Consider a system of two types of particles, A (catalyst) and B (reactant), subject to:

- A-particles evolve autonomously according to a prescribed dynamics with $\xi(x,t)$ denoting the number of A-particles at site x at time t;
- B-particles perform independent random walks at rate $2d\kappa$ and split into two at a rate that is equal to γ times the number of A-particles present at the same location;
- B-particles die at rate δ;
- the initial configuration of B-particles is one particle at site 0 and no particle elsewhere.

Then

$$u(x,t) = \begin{array}{l}\text{the average number of } B\text{-particles at site } x \text{ at time } t \\ \text{conditioned on the evolution of the } A\text{-particles.}\end{array} \tag{5}$$

It is possible to remove δ via the trivial transformation $u(x,t) \to u(x,t)e^{-\delta t}$. In what follows we will therefore put $\delta = 0$.

Throughout the paper, $\mathbb{P}$ denotes the law of ξ and we assume that

$$\bullet \quad \begin{array}{l} \xi \text{ is } \textit{stationary} \text{ and } \textit{ergodic} \text{ under translations in space and time,} \\ \xi \text{ is } \textit{not constant} \text{ and } \rho = \mathbb{E}(\xi(0,0)) \in \mathbb{R}, \end{array} \tag{6}$$

and

$$\bullet \quad \forall\, \kappa, \gamma \in [0,\infty)\ \exists\, c = c(\kappa,\gamma) < \infty\colon\ \mathbb{E}(\log u(0,t)) \leq ct\ \forall\, t \geq 0. \tag{7}$$

Three choices of ξ will receive special attention:

(1) *Independent Simple Random Walks* (ISRW) [Kipnis and Landim [22], Chapter 1]. Here, $\xi_t \in \Omega = (\mathbb{N} \cup \{0\})^{\mathbb{Z}^d}$ and $\xi(x,t)$ represents the number of particles at site x at time t. Under the ISRW-dynamics particles move around independently as simple random walks stepping at rate 1. We draw ξ_0 according to the equilibrium ν_ρ with density $\rho \in (0,\infty)$, which is a Poisson product measure.

(2) *Symmetric Exclusion Process* (SEP) [Liggett [23], Chapter VIII]. Here, $\xi_t \in \Omega = \{0,1\}^{\mathbb{Z}^d}$ and $\xi(x,t)$ represents the presence ($\xi(x,t) = 1$) or absence ($\xi(x,t) = 0$) of a particle at site x at time t. Under the SEP-dynamics particles move around independently according to an irreducible symmetric random walk transition kernel at rate 1, but subject to the restriction that no two particles can occupy the same site. We draw ξ_0 according to the equilibrium ν_ρ with density $\rho \in (0,1)$, which is a Bernoulli product measure.

(3) *Symmetric Voter Model* (SVM) [Liggett [23], Chapter V]. Here, $\xi_t \in \Omega = \{0,1\}^{\mathbb{Z}^d}$ and $\xi(x,t)$ represents the opinion of a voter at site x at time t. Under the SVM-dynamics each voter imposes its opinion on another voter according to an irreducible symmetric random walk transition kernel at rate 1. We draw ξ_0 according to the equilibrium distribution ν_ρ with density $\rho \in (0,1)$, which is not a product measure.

Note: While ISRW and SEP are conservative and reversible in time, SVM is not. The equilibrium properties of SVM are qualitatively different for recurrent and transient random walk. For *recurrent* random walk all equilibria with $\rho \in (0,1)$ are non-ergodic, namely, $\nu_\rho = (1-\rho)\delta_{\{\eta\equiv 0\}} + \rho\,\delta_{\{\eta\equiv 1\}}$, and therefore are precluded by (6). For *transient* random walk, on the other hand, there are ergodic equilibria.

1.2 Lyapunov exponents

Our focus will be on the *quenched* Lyapunov exponent, defined by

$$\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log u(0,t) \qquad \xi\text{-a.s.} \tag{8}$$

We will be interested in comparing λ_0 with the *annealed* Lyapunov exponents, defined by

$$\lambda_p = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big([u(0,t)]^p\big)^{1/p}, \qquad p \in \mathbb{N}, \tag{9}$$

which were analyzed in detail in our earlier work (see Section 1.3). In (8–9) we pick $x=0$ as the reference site to monitor the growth of u. However, it is easy to show that the Lyapunov exponents are the same at other sites.

By the Feynman-Kac formula, the solution of (1) reads

$$u(x,t) = \mathrm{E}_x\left(\exp\left[\gamma\int_0^t \xi\,(X^\kappa(s),t-s)\,ds\right] u\big(X^\kappa(t),0\big)\right), \tag{10}$$

where $X^\kappa = (X^\kappa(t))_{t\geq 0}$ is simple random walk on $\mathbb{Z}^d$ stepping at rate $2d\kappa$ and E_x denotes expectation with respect to X^κ given $X^\kappa(0)=x$. In particular, for our choice in (4), for any $t>0$ we have

$$\begin{aligned} u(0,t) &= \mathrm{E}_0\left(\exp\left[\gamma\int_0^t \xi\big(X^\kappa(s),t-s\big)\,ds\right]\delta_0\big(X^\kappa(t)\big)\right)\\ &= \mathrm{E}_0\left(\exp\left[\gamma\int_0^t \xi\big(X^\kappa(s),s\big)\,ds\right]\delta_0\big(X^\kappa(t)\big)\right), \end{aligned} \tag{11}$$

where in the last line we reverse time and use that X^κ is a reversible dynamics. Therefore, we can define

$$\Lambda_0(t) = \frac{1}{t}\log u(0,t) = \frac{1}{t}\log \mathrm{E}_0\left(\exp\left[\gamma\int_0^t \xi\big(X^\kappa(s),s\big)\,ds\right] u\big(X^\kappa(t),0\big)\right). \tag{12}$$

If the last quantity ξ-a.s. admits a limit as $t\to\infty$, then

$$\lambda_0 = \lim_{t\to\infty} \Lambda_0(t) \qquad \xi\text{-a.s.}, \tag{13}$$

where the limit is expected to be ξ-a.s. constant.

Clearly, λ_0 is a function of d, κ, γ and the parameters controlling ξ. In what follows, our main focus will be on the dependence on κ, and therefore we will often write $\lambda_0(\kappa)$. Note that $p\mapsto\lambda_p(\kappa)$ is non-decreasing for $p\in\mathbb{N}\cup\{0\}$.

Note: Conditions (6–7) imply that the expectations in (10–12) are strictly positive and finite for all $x\in\mathbb{Z}^d$ and $t\geq 0$, and that $\lambda_0<\infty$. Moreover, by Jensen's inequality applied to (12) with $u(\cdot,0)$ given by (4), we have $\mathbb{E}(\Lambda_0(t)) \geq \rho\gamma + \frac{1}{t}\log \mathrm{P}_0(X^\kappa(t)=0)$ and, since the last term tends to zero as $t\to\infty$, we find that $\lambda_0\geq\rho\gamma>-\infty$.

1.3 Literature

The behavior of the Lyapunov exponents for the PAM in a *time-dependent* random environment has been the subject of several papers.

1.3.1 White noise

Carmona and Molchanov [6] obtained a qualitative description of both the *quenched* and the *annealed* Lyapunov exponents when ξ is white noise, i.e.,

$$\xi(x,t) = \frac{\partial}{\partial t} W(x,t), \tag{14}$$

where $W = (W_t)_{t\geq 0}$ with $W_t = \{W(x,t)\colon\ x \in \mathbb{Z}^d\}$ is a space-time field of independent Brownian motions. This choice is special because the increments of ξ are *independent in space and time*. They showed that if $u(\cdot,0)$ has compact support (e.g. $u(\cdot,0) = \delta_0(\cdot)$ as in (4)), then the quenched Lyapunov exponent $\lambda_0(\kappa)$ defined in (8) exists and is constant ξ-a.s., and is independent of $u(\cdot,0)$. Moreover, they found that the asymptotics of $\lambda_0(\kappa)$ as $\kappa \downarrow 0$ is singular, namely, there are constants $C_1, C_2 \in (0,\infty)$ and $\kappa_0 \in (0,\infty)$ such that

$$C_1 \frac{1}{\log(1/\kappa)} \leq \lambda_0(\kappa) \leq C_2 \frac{\log\log(1/\kappa)}{\log(1/\kappa)} \qquad \forall\, 0 < \kappa \leq \kappa_0. \tag{15}$$

Subsequently, Carmona, Molchanov and Viens [7], Carmona, Koralov and Molchanov [5], and Cranston, Mountford and Shiga [9], proved the existence of λ_0 when $u(\cdot,0)$ has non-compact support (e.g. $u(\cdot,0) \equiv 1$), showed that there is a constant $C \in (0,\infty)$ such that

$$\lim_{\kappa\downarrow 0} \log(1/\kappa)\, \lambda_0(\kappa) = C, \tag{16}$$

and proved that

$$\lim_{p\downarrow 0} \lambda_p(\kappa) = \lambda_0(\kappa) \quad \forall\, \kappa \in [0,\infty). \tag{17}$$

(These results were later extended to Lévy white noise by Cranston, Mountford and Shiga [10], and to colored noise by Kim, Viens and Vizcarra [20].) Further refinements on the behavior of the Lyapunov exponents were conjectured in Carmona and Molchanov [6] and proved in Greven and den Hollander [18]. In particular, it was shown that $\lambda_1(\kappa) = \frac{1}{2}$ for all $\kappa \in [0,\infty)$, while for the other Lyapunov exponents the following dichotomy holds (see Figs. 1–2):

- $d = 1,2$: $\lambda_0(\kappa) < \frac{1}{2}$, $\lambda_p(\kappa) > \frac{1}{2}$ for $p \in \mathbb{N}\backslash\{1\}$, for $\kappa \in [0,\infty)$;
- $d \geq 3$: there exist $0 < \kappa_0 \leq \kappa_2 \leq \kappa_3 \leq \ldots < \infty$ such that

$$\lambda_0(\kappa) - \tfrac{1}{2} \begin{cases} < 0, \text{ for } \kappa \in [0,\kappa_0), \\ = 0, \text{ for } \kappa \in [\kappa_0,\infty), \end{cases} \tag{18}$$

and

$$\lambda_p(\kappa) - \tfrac{1}{2} \begin{cases} > 0, \text{ for } \kappa \in [0, \kappa_p), \\ = 0, \text{ for } \kappa \in [\kappa_p, \infty), \end{cases} \qquad p \in \mathbb{N}\backslash\{1\}. \tag{19}$$

Moreover, variational formulas for κ_p were derived, which in turn led to upper and lower bounds on κ_p, and to the identification of the asymptotics of κ_p for $p \to \infty$ (κ_p grows linearly with p). In addition, it was shown that for every $p \in \mathbb{N}\backslash\{1\}$ there exists a $d(p) < \infty$ such that $\kappa_p < \kappa_{p+1}$ for $d \geq d(p)$. Moreover, it was shown that $\kappa_0 < \kappa_2$ in Birkner, Greven and den Hollander [2] ($d \geq 5$), Birkner and Sun [3] ($d = 4$), Berger and Toninelli [1], Birkner and Sun [4] ($d = 3$). Note that, by Hölder's inequality, all curves in Figs. 1–2 are distinct whenever they are different from $\frac{1}{2}$.

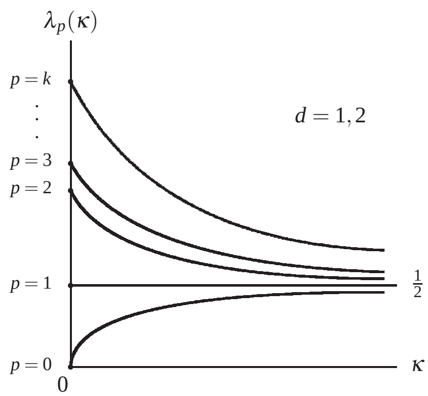

Fig. 1 Quenched and annealed Lyapunov exponents when $d = 1,2$ for white noise.

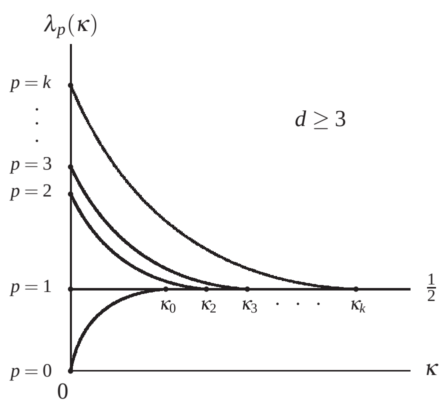

Fig. 2 Quenched and annealed Lyapunov exponents when $d \geq 3$ for white noise.

1.3.2 Interacting particle systems

Various models where ξ is *dependent in space and time* were looked at subsequently. Kesten and Sidoravicius [19], and Gärtner and den Hollander [13], considered the case where ξ is a field of independent simple random walks in Poisson equilibrium (ISRW). The survival versus extinction pattern [19] and the annealed Lyapunov exponents [13] were analyzed, in particular, their dependence on d, κ, γ and ρ. The case where ξ is a single random walk was studied by Gärtner and Heydenreich [12]. Gärtner, den Hollander and Maillard [14], [16], [17] subsequently considered the cases where ξ is an exclusion process with an irreducible symmetric random walk transition kernel starting from a Bernoulli product measure (SEP), respectively, a voter model with an irreducible symmetric *transient* random walk transition kernel starting either from a Bernoulli product measure or from equilibrium (SVM). In each of these cases, a fairly complete picture of the behavior of the annealed Lyapunov exponents was obtained, including the presence or absence of *intermittency*, i.e., $\lambda_p(\kappa) > \lambda_{p-1}(\kappa)$ for some or all values of $p \in \mathbb{N}\backslash\{1\}$ and $\kappa \in [0,\infty)$. Several conjectures were formulated as well. In what follows we describe these results in some more detail. We refer the reader to Gärtner, den Hollander and Maillard [15] for an overview.

It was shown in Gärtner and den Hollander [13], and Gärtner, den Hollander and Maillard [14], [16], [17] that for ISRW, SEP and SVM in equilibrium the function $\kappa \mapsto \lambda_p(\kappa)$ satisfies:

- If $d \geq 1$ and $p \in \mathbb{N}$, then the limit in (9) exists for all $\kappa \in [0,\infty)$. Moreover, if $\lambda_p(0) < \infty$, then $\kappa \mapsto \lambda_p(\kappa)$ is finite, continuous, strictly decreasing and convex on $[0,\infty)$.
- There are two regimes (we summarize results only for the case where the random walk transition kernel has finite second moment):
  - *Strongly catalytic regime* (see Fig. 3):
    - ISRW: $d = 1,2$, $p \in \mathbb{N}$ or $d \geq 3$, $p \geq 1/\gamma G_d$: $\lambda_p \equiv \infty$ on $[0,\infty)$. (G_d is the Green function at the origin of simple random walk.)
    - SEP: $d = 1,2$, $p \in \mathbb{N}$: $\lambda_p \equiv \gamma$ on $[0,\infty)$.
    - SVM: $d = 1,2,3,4$, $p \in \mathbb{N}$: $\lambda_p \equiv \gamma$ on $[0,\infty)$.
  - *Weakly catalytic regime* (see Fig. 4–5):
    - ISRW: $d \geq 3$, $p < 1/\gamma G_d$: $\rho\gamma < \lambda_p < \infty$ on $[0,\infty)$.
    - SEP: $d \geq 3$, $p \in \mathbb{N}$: $\rho\gamma < \lambda_p < \gamma$ on $[0,\infty)$.
    - SVM: $d \geq 5$, $p \in \mathbb{N}$: $\rho\gamma < \lambda_p < \gamma$ on $[0,\infty)$.
- For all three dynamics, in the weakly catalytic regime $\lim_{\kappa\to\infty} \kappa[\lambda_p(\kappa) - \rho\gamma] = C_1 + C_2 p^2 1_{\{d=d_c\}}$ with $C_1, C_2 \in (0,\infty)$ and d_c a critical dimension: $d_c = 3$ for ISRW, SEP and $d_c = 5$ for SVM.
- Intermittent behavior:
  - In the strongly catalytic regime, there is no intermittency for all three dynamics.
  - In the weakly catalytic regime, there is full intermittency for:

- all three dynamics when $0 \le \kappa \ll 1$.
- ISRW and SEP in $d = 3$ when $\kappa \gg 1$.
- SVM in $d = 5$ when $\kappa \gg 1$.

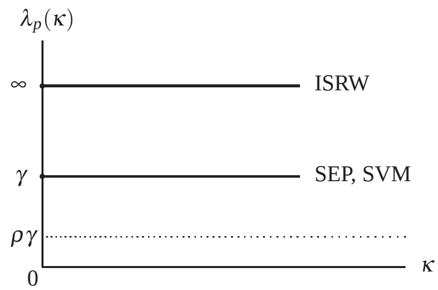

Fig. 3 Triviality of the annealed Lyapunov exponents for ISRW, SEP, SVM in the strongly catalytic regime.

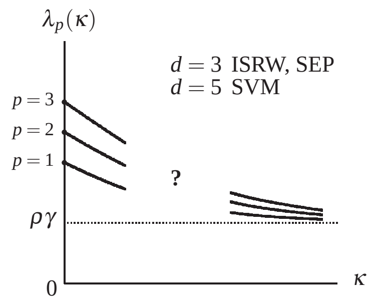

Fig. 4 Non-triviality of the annealed Lyapunov exponents for ISRW, SEP and SVM in the weakly catalytic regime at the critical dimension.

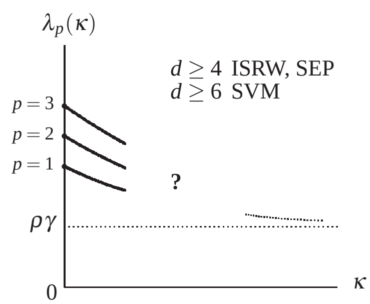

Fig. 5 Non-triviality of the annealed Lyapunov exponents for ISRW, SEP and SVM in the weakly catalytic regime above the critical dimension.

Note: For SVM the convexity of $\kappa \mapsto \lambda_p(\kappa)$ and its scaling behavior for $\kappa \to \infty$ have not actually been proved, but have been argued on heuristic grounds.

Recently, there has been further progress for the case where ξ consists of 1 random walk (Schnitzler and Wolff [25]) or n independent random walks (Castell, Gün and Maillard [8]), ξ is the SVM (Maillard, Mountford and Schöpfer [24]), and for the trapping version of the PAM with $\gamma \in (-\infty, 0)$ (Drewitz, Gärtner, Ramírez and Sun [11]). All these papers appear elsewhere in the present volume.

1.4 Main results

We have six theorems, all relating to the *quenched* Lyapunov exponent and extending the results on the annealed Lyapunov exponents listed in Section 1.3.

Let e be any nearest-neighbor site of 0, and abbreviate

$$I^\xi(x,t) = \int_0^t [\xi(x,s) - \rho]\, ds, \qquad x \in \mathbb{Z}^d,\, t \geq 0. \tag{20}$$

Our first three theorems deal with general ξ and employ four successively stronger *noisiness conditions*:

$$\lim_{t\to\infty} \frac{1}{\log t}\, \mathbb{E}\big(|I^\xi(0,t) - I^\xi(e,t)|\big) = \infty, \tag{21}$$

$$\liminf_{t\to\infty} \frac{1}{t}\, \mathbb{E}\big(|I^\xi(0,t) - I^\xi(e,t)|^2\big) > 0, \ \limsup_{t\to\infty} \frac{1}{t^2}\, \mathbb{E}\big(|I^\xi(0,t)|^4\big) < \infty, \tag{22}$$

$$\limsup_{t\to\infty} \frac{1}{t^{2/3}} \log\Big[\sup_{\eta\in\Omega} \mathbb{P}_\eta\big(I^\xi(0,t) > t^{5/6}\big)\Big] < 0, \tag{23}$$

$$\exists c < \infty\colon \ \sup_{\eta\in\Omega} \mathbb{E}_\eta\big(\exp\big[\mu I^\xi(0,t)\big]\big) \leq \exp[c\mu^2 t] \qquad \forall\, \mu, t > 0, \tag{24}$$

where $\mathbb{P}_\eta$ denotes the law of ξ starting from $\xi_0 = \eta$.

Theorem 1.1. *Fix $d \geq 1$, $\kappa \in [0,\infty)$ and $\gamma \in (0,\infty)$. The limit in* (8) *exists $\mathbb{P}$-a.s. and in $\mathbb{P}$-mean, and is finite.*

Theorem 1.2. *Fix $d \geq 1$ and $\gamma \in (0,\infty)$.*
(i) $\lambda_0(0) = \rho\gamma$ and $\rho\gamma < \lambda_0(\kappa) < \infty$ for all $\kappa \in (0,\infty)$ with $\rho = \mathbb{E}(\xi(0,0)) \in \mathbb{R}$.
(ii) $\kappa \mapsto \lambda_0(\kappa)$ is globally Lipschitz outside any neighborhood of 0. Moreover, if ξ is bounded from above, then the Lipschitz constant at κ tends to zero as $\kappa \to \infty$.
(iii) If ξ satisfies condition (21) *and is bounded from below, then $\kappa \mapsto \lambda_0(\kappa)$ is not Lipschitz at 0.*

Theorem 1.3. *(i) If ξ satisfies condition* (22) *and is bounded from below, then*

$$\liminf_{\kappa\downarrow 0} \log(1/\kappa)\, [\lambda_0(\kappa) - \rho\gamma] > 0. \tag{25}$$

(ii) If ξ is a Markov process that satisfies condition (23) *and is bounded from above, then*

$$\limsup_{\kappa\downarrow 0}\,[\log(1/\kappa)]^{1/6}\,[\lambda_0(\kappa)-\rho\gamma]<\infty. \tag{26}$$

(iii) If ξ is a Markov process that satisfies condition (24) *and is bounded from above, then*

$$\limsup_{\kappa\downarrow 0}\,\frac{\log(1/\kappa)}{\log\log(1/\kappa)}\,[\lambda_0(\kappa)-\rho\gamma]<\infty. \tag{27}$$

Our last three theorems deal with ISRW, SEP and SVM.

Theorem 1.4. *For ISRW, SEP and SVM in the weakly catalytic regime,* $\lim_{\kappa\to\infty}\lambda_0(\kappa)=\rho\gamma$*.*

Theorem 1.5. *ISRW and SEP satisfy conditions* (21) *and* (22)*.*

Theorem 1.6. *For ISRW in the strongly catalytic regime, $\lambda_0(\kappa)<\lambda_1(\kappa)$ for all $\kappa\in[0,\infty)$.*

Theorems 1.1–1.3 wil be proved in Section 2, Theorems 1.4–1.6 in Section 3. *Note:* Theorem 1.4 extends to voter models that are non necessarily symmetric (see Section 3.1).

1.5 Discussion and open problems

1. Fig. 6 graphically summarizes the results in Theorems 1.1–1.3 and what we expect to be provable with al little more effort. The main message of this figure is that the *qualitative* behavior of $\kappa\mapsto\lambda_0(\kappa)$ is well understood, including the logarithmic singularity at $\kappa=0$. Note that Theorems 1.2 and 1.3(i) do not imply continuity at $\kappa=0$, while Theorems 1.3(ii–iii) do.

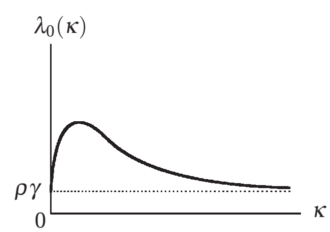

Fig. 6 Conjectured behavior of the quenched Lyapunov exponent.

2. Figs. 7–9 summarize how we expect $\kappa\mapsto\lambda_0(\kappa)$ to compare with $\kappa\mapsto\lambda_1(\kappa)$ for the three dynamics.

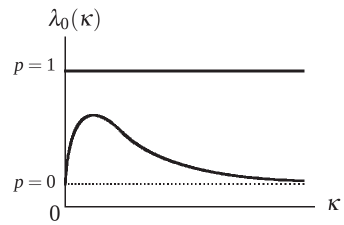

Fig. 7 Conjectured behavior for ISRW, SEP and SVM below the critical dimension.

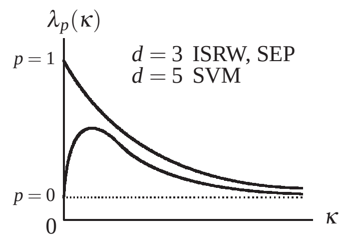

Fig. 8 Conjectured behavior for ISRW, SEP and SVM at the critical dimension.

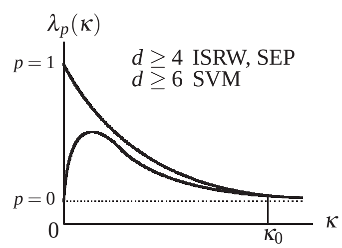

Fig. 9 Conjectured behavior for ISRW, SEP and SVM above the critical dimension.

3. Conditions (6–7) are trivially satisfied for SEP and SVM, because ξ is bounded. For ISRW they follow from Kesten and Sidoravicius [19], Theorem 2.

4. Conditions (21–22) are *weak* while conditions (23–24) are *strong*. Theorem 1.5 states that conditions (21–22) are satisfied for ISRW and SEP. We will see in Section 3.2 that, most likely, they are satisfied for SVM as well. Conditions (23–24) fail for the three dynamics, but are satisfied e.g. for spin-flip dynamics in the so-called "$M<\varepsilon$ regime" (see Liggett [23], Section I.3). [The verification of this statement is left to the reader.]

5. The following problems remain open:

- Extend Theorem 1.1 to the initial condition $u(\cdot,0)\equiv 1$, and show that λ_0 is the same as for the initial condition $u(\cdot,0)=\delta_0(\cdot)$ assumed in (4). [The proof of

Theorem 1.1 in Section 2.1 shows that it is straightforward to do this extension for $u(\cdot,0)$ symmetric with bounded support. Recall the remark made prior to (12).]

- Prove that $\lim_{\kappa\downarrow 0}\lambda_0(\kappa)=\rho\gamma$ and $\lim_{\kappa\to\infty}\lambda_0(\kappa)=\rho\gamma$ under conditions (6–7) alone. [These limits correspond to time ergodicity, respectively, space ergodicity of ξ, but are non-trivial because they require some control on the fluctuations of ξ.]
- Prove Theorems 1.2(ii–iii) without the boundedness assumptions on ξ. Prove Theorem 1.3(i) under condition (21) alone. [The proof of Theorem 1.2(iii) in Section 2.4 shows that $\lambda_0(\kappa)-\rho\gamma$ stays above any positive power of κ as $\kappa\downarrow 0$.] Improve Theorems 1.3(ii–iii) by establishing under what conditions the upper bounds in (26–27) can be made to match the lower bound in (25).
- Extend Theorems 1.4–1.6 by proving the qualitative behavior for the three dynamics conjectured in Figs. 7–9. [For white noise dynamics the curves successively merge for all $d\geq 3$ (see Figs. 1–2).]
- For the three dynamics in the weakly catalytic regime, find the asymptotics of $\lambda_0(\kappa)$ as $\kappa\to\infty$ and compare this with the asymptotics of $\lambda_p(\kappa)$, $p\in\mathbb{N}$, as $\kappa\to\infty$ (see Figs. 4–5).
- Extend the existence of λ_p to all (non-integer) $p>0$, and prove that $\lambda_p\downarrow\lambda_0$ as $p\downarrow 0$. [For white noise dynamics this extension is achieved in (17).]

2 Proof of Theorems 1.1–1.3

The proofs of Theorems 1.1–1.3 are given in Sections 2.1, 2.2–2.4 and 2.5–2.7, respectively. W.l.o.g. we may assume that $\rho=\mathbb{E}(\xi(0,0))=0$, by the remark made prior to conditions (6–7).

2.1 Proof of Theorem 1.1

Proof. Recall (4) and (12–13), abbreviate

$$\chi(s,t)=\mathrm{E}_0\left(\exp\left[\gamma\int_0^{t-s}\xi\left(X^\kappa(v),s+v\right)dv\right]\delta_0(X^\kappa(t-s))\right),\qquad 0\leq s\leq t<\infty, \tag{28}$$

and note that $\chi(0,t)\stackrel{\mathbb{P}}{=}u(0,t)$. Picking $u\in[s,t]$, inserting $\delta_0(X^\kappa(u-s))$ under the expectation and using the Markov property of X^κ at time $u-s$, we obtain

$$\chi(s,t)\geq\chi(s,u)\,\chi(u,t),\qquad 0\leq s\leq u\leq t<\infty. \tag{29}$$

Thus, $(s,t)\mapsto\log\chi(s,t)$ is superadditive. By condition (6), the law of $\{\chi(u+s,u+t)\colon 0\leq s\leq t<\infty\}$ is the same for all $u\geq 0$. Therefore the superadditive ergodic

theorem (see Kingman [21]) implies that

$$\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log \chi(0,t) \text{ exists } \mathbb{P}\text{-a.s. and in } \mathbb{P}\text{-mean.} \tag{30}$$

We saw at the end of Section 1.2 that $\lambda_0 \in [0,\infty)$ (because $\rho = 0$). $\square$

2.2 Proof of Theorem 1.2(i)

Proof. The fact that $\lambda_0(0) = 0$ is immediate from (12–13) because $\mathrm{P}_0(X^0(t) = 0) = 1$ for all $t \geq 0$ and $\int_0^t \xi(0,s)\,ds = o(t)$ ξ-a.s. as $t \to \infty$ by the ergodic theorem (recall condition (6)). We already know that $\lambda_0(\kappa) \in [0,\infty)$ for all $\kappa \in [0,\infty)$. The proof of the strict lower bound $\lambda_0(\kappa) > 0$ for $\kappa \in (0,\infty)$ comes in 2 steps.

1. Fix $T > 0$ and consider the expression

$$\lambda_0 = \lim_{n\to\infty} \frac{1}{nT} \mathbb{E}\big(\log u(0,nT)\big) \qquad \xi\text{-a.s.} \tag{31}$$

Partition the time interval $[0,nT)$ into n pieces of length T,

$$\mathscr{I}_j = [(j-1)T, jT), \qquad j = 1,\dots,n. \tag{32}$$

Use the Markov property of X^κ at the beginning of each piece, to obtain

$$\begin{aligned} &u(0,nT) \\ &= \mathrm{E}_0\left(\exp\left[\gamma \sum_{j=1}^n \int_{\mathscr{I}_j} \xi\big(X^\kappa(s),s\big)\,ds\right] \delta_0(X^\kappa(nT))\right) \\ &= \sum_{x_1,\dots,x_{n-1}\in\mathbb{Z}^d} \prod_{j=1}^n \mathrm{E}_{x_{j-1}}\left(\exp\left[\gamma \int_0^T \xi\big(X^\kappa(s),(j-1)T+s\big)\,ds\right] \delta_{x_j}(X^\kappa(T))\right) \end{aligned} \tag{33}$$

with $x_0 = x_n = 0$. Next, for $x,y \in \mathbb{Z}^d$, let $\mathrm{E}_{x,y}^{(T)}$ denote the conditional expectation over X^κ given that $X^\kappa(0) = x$ and $X^\kappa(T) = y$, and abbreviate, for $1 \leq j \leq n$,

$$\mathrm{E}_{x,y}^{(T)}(j) = \mathrm{E}_{x,y}^{(T)}\left(\exp\left[\gamma \int_0^T \xi\big(X^\kappa(s),(j-1)T+s\big)\,ds\right]\right). \tag{34}$$

Then we can write

$$\begin{aligned} &\mathrm{E}_{x_{j-1}}\left(\exp\left[\gamma \int_0^T \xi\big(X^\kappa(s),(j-1)T+s\big)\,ds\right] \delta_{x_j}(X^\kappa(T))\right) \\ &\qquad = p_T^\kappa(x_j - x_{j-1})\, \mathrm{E}_{x_{j-1},x_j}^{(T)}(j), \end{aligned} \tag{35}$$

where we abbreviate $p_T^\kappa(x) = \mathbb{P}_0(X^\kappa(T) = x)$, $x \in \mathbb{Z}^d$. Combined with (33), this gives

$$\begin{aligned} u(0,nT) &= \sum_{x_1,\cdots,x_{n-1}\in\mathbb{Z}^d} \left(\prod_{j=1}^n p_T^\kappa(x_j - x_{j-1})\right) \left(\prod_{j=1}^n \mathrm{E}^{(T)}_{x_{j-1},x_j}(j)\right) \\ &= p_{nT}^\kappa(0)\, \mathrm{E}_{0,0}^{(nT)} \left(\prod_{j=1}^n \mathrm{E}^{(T)}_{X^\kappa((j-1)T),X^\kappa(jT)}(j)\right). \end{aligned} \tag{36}$$

2. To estimate the last expectation in (36), abbreviate $\xi_I = (\xi_t)_{t\in I}$, $I \subset [0,\infty)$, and apply Jensen's inequality to (34), to obtain

$$\mathrm{E}^{(T)}_{x,y}(j) = \exp\left[\gamma \int_0^T \mathrm{E}^{(T)}_{x,y}\Big(\xi\big(X^\kappa(s),(j-1)T+s\big)\Big)\,ds + C_{x,y}\big(\xi_{[(j-1)T,jT]},T\big)\right] \tag{37}$$

for some $C_{x,y}(\xi_{[(j-1)T,jT]},T)$ that satisfies

$$C_{x,y}\big(\xi_{[(j-1)T,jT]},T\big) > 0 \quad \xi\text{-a.s.} \qquad \forall x,y \in \mathbb{Z}^d,\, 1 \le j \le n. \tag{38}$$

Here, the strict positivity is an immediate consequence of the fact that ξ is not constant (recall condition (6)) and $u \mapsto e^u$ is strictly convex. Combining (36–37) and using Jensen's inequality again, this time w.r.t. $\mathrm{E}_{0,0}^{(nT)}$, we obtain

$$\begin{aligned} &\mathbb{E}\Big(\log u(0,nT)\Big) \\ &\ge \log p_{nT}^\kappa(0) \\ &\quad + \mathbb{E}\left(\mathrm{E}_{0,0}^{(nT)}\left(\sum_{j=1}^n \mathrm{E}^{(T)}_{X^\kappa((j-1)T),X^\kappa(jT)}\left(\gamma\int_0^T \xi\big(X^\kappa(s),(j-1)T+s\big)\,ds\right.\right.\right. \\ &\qquad\qquad \left.\left.\left. + C_{X^\kappa((j-1)T),X^\kappa(jT)}\big(\xi_{[(j-1)T,jT]},T\big)\right)\right)\right) \\ &= \log p_{nT}^\kappa(0) \\ &\quad + \mathbb{E}\left(\mathrm{E}_{0,0}^{(nT)}\left(\sum_{j=1}^n \mathrm{E}^{(T)}_{X^\kappa((j-1)T),X^\kappa(jT)}\Big(C_{X^\kappa((j-1)T),X^\kappa(jT)}\big(\xi_{[(j-1)T,jT]},T\big)\Big)\right)\right), \end{aligned} \tag{39}$$

where the middle term in the second line vanishes because of condition (6) and our assumption that $\mathbb{E}(\xi(0,0)) = 0$. After inserting the indicator of the event $\{X^\kappa((j-1)T) = X^\kappa(jT)\}$ for $1 \le j \le n$ in the last expectation in (39), we get

$$
\begin{aligned}
&\mathbb{E}\Bigg(\mathrm{E}_{0,0}^{(nT)}\Bigg(\sum_{j=1}^{n}\mathrm{E}^{(T)}_{X^\kappa((j-1)T),X^\kappa(jT)}\Big(C_{X^\kappa((j-1)T),X^\kappa(jT)}\big(\xi_{[(j-1)T,jT]},T\big)\Big)\Bigg)\Bigg)\\
&\quad\geq \sum_{j=1}^{n}\sum_{z\in\mathbb{Z}^d}\frac{p^\kappa_{(j-1)T}(z)\,p^\kappa_T(0)\,p^\kappa_{(n-j)T}(z)}{p^\kappa_{nT}(0)}\,\mathbb{E}\Big(C_{z,z}\big(\xi_{[(j-1)T,jT]},T\big)\Big)\\
&\quad\geq nC_T\,p^\kappa_T(0),
\end{aligned}
\tag{40}
$$

where we abbreviate

$$
C_T=\mathbb{E}\Big(C_{z,z}\big(\xi_{[(j-1)T,jT]},T\big)\Big)>0,
\tag{41}
$$

note that the latter does not depend on j or z, and use that $\sum_{z\in\mathbb{Z}^d}p^\kappa_{(j-1)T}(z)p^\kappa_{(n-j)T}(z)$ $=p^\kappa_{(n-1)T}(0)\geq p^\kappa_{nT}(0)$. Therefore, combining (31) and (39–41), and using that

$$
\lim_{n\to\infty}\frac{1}{nT}\log p^\kappa_{nT}(0)=0,
\tag{42}
$$

we arrive at $\lambda_0\geq (C_T/T)p^\kappa_T(0)>0$. $\square$

2.3 Proof of Theorem 1.2(ii)

Proof. In Step 1 we prove the Lischitz continuity outside any neighborhood of 0 under the restriction that $\xi\leq 1$. This proof is essentially a copy of the proof in Gärtner, den Hollander and Maillard [17] of the Lipschitz continuity of the annealed Lyapunov exponents when ξ is SVM. In Step 2 we explain how to remove the restriction $\xi\leq 1$. In Step 3 we show that the Lipschitz constant tends to zero as $\kappa\to\infty$ when $\xi\leq 1$.

1. Pick $\kappa_1,\kappa_2\in(0,\infty)$ with $\kappa_1<\kappa_2$ arbitrarily. By Girsanov's formula,

$$
\begin{aligned}
&\mathrm{E}_0\bigg(\exp\Big[\gamma\int_0^t\xi(X^{\kappa_2}(s),s)\,ds\Big]\,\delta_0(X^{\kappa_2}(t))\bigg)\\
&\quad=\mathrm{E}_0\bigg(\exp\Big[\gamma\int_0^t\xi(X^{\kappa_1}(s),s)\,ds\Big]\,\delta_0(X^{\kappa_1}(t))\\
&\qquad\qquad\times\exp\Big[J(X^{\kappa_1};t)\log(\kappa_2/\kappa_1)-2d(\kappa_2-\kappa_1)t\Big]\bigg)\\
&\quad=I+II,
\end{aligned}
\tag{43}
$$

where $J(X^{\kappa_1};t)$ is the number of jumps of X^{κ_1} up to time t, I and II are the contributions coming from the events $\{J(X^{\kappa_1};t)\leq M2d\kappa_2 t\}$, respectively, $\{J(X^{\kappa_1};t)> M2d\kappa_2 t\}$, with $M>1$ to be chosen. Clearly,

$$
\begin{aligned}
I \le \exp\Big[\Big(M2d\kappa_2\log(\kappa_2/\kappa_1) - 2d(\kappa_2-\kappa_1)\Big)t\Big] \\
\times \mathrm{E}_0\bigg(\exp\bigg[\gamma\int_0^t \xi(X^{\kappa_1}(s),s)\,ds\bigg]\,\delta_0(X^{\kappa_1}(t))\bigg),
\end{aligned} \tag{44}
$$

while

$$
II \le e^{\gamma t}\,\mathrm{P}_0\Big(J(X^{\kappa_2};t) > M2d\kappa_2 t\Big) \tag{45}
$$

because we may estimate

$$
\int_0^t \xi(X^{\kappa_1}(s),s)\,ds \le t \tag{46}
$$

and afterwards use Girsanov's formula in the reverse direction. Since $J(X^{\kappa_2};t) = J^*(2d\kappa_2 t)$ with $(J^*(t))_{t\ge 0}$ a rate-1 Poisson process, we have

$$
\lim_{t\to\infty}\frac{1}{t}\log\mathrm{P}_0\Big(J(X^{\kappa_2};t) > M2d\kappa_2 t\Big) = -2d\kappa_2\mathscr{I}(M) \tag{47}
$$

with

$$
\mathscr{I}(M) = \sup_{u\in\mathbb{R}}\big[Mu - \big(e^u - 1\big)\big] = M\log M - M + 1. \tag{48}
$$

Recalling (12–13), we get from (43–47) the upper bound

$$
\lambda_0(\kappa_2) \le \big[M2d\kappa_2\log(\kappa_2/\kappa_1) - 2d(\kappa_2-\kappa_1) + \lambda_0(\kappa_1)\big] \vee \big[\gamma - 2d\kappa_2\mathscr{I}(M)\big]. \tag{49}
$$

On the other hand, estimating $J(X^{\kappa_1};t) \ge 0$ in (43), we have

$$
\begin{aligned}
&\mathrm{E}_0\bigg(\exp\bigg[\gamma\int_0^t \xi(X^{\kappa_2}(s),s)\,ds\bigg]\,\delta_0(X^{\kappa_2}(t))\bigg) \\
&\qquad \ge \exp[-2d(\kappa_2-\kappa_1)t]\;\mathrm{E}_0\bigg(\exp\bigg[\gamma\int_0^t \xi(X^{\kappa_1}(s),s)\,ds\bigg]\,\delta_0(X^{\kappa_1}(t))\bigg),
\end{aligned} \tag{50}
$$

which gives the lower bound

$$
\lambda_0(\kappa_2) \ge -2d(\kappa_2-\kappa_1) + \lambda_0(\kappa_1). \tag{51}
$$

Next, for $\kappa \in (0,\infty)$, define

$$
\begin{aligned}
D^+\lambda_0(\kappa) &= \limsup_{\varepsilon\to 0}\,\varepsilon^{-1}[\lambda_0(\kappa+\varepsilon) - \lambda_0(\kappa)], \\
D^-\lambda_0(\kappa) &= \liminf_{\varepsilon\to 0}\,\varepsilon^{-1}[\lambda_0(\kappa+\varepsilon) - \lambda_0(\kappa)],
\end{aligned} \tag{52}
$$

where $\varepsilon \to 0$ from both sides. Then, in (49) and (51), picking $\kappa_1 = \kappa$ and $\kappa_2 = \kappa+\delta$, respectively, $\kappa_1 = \kappa-\delta$ and $\kappa_2 = \kappa$ with $\delta > 0$ and letting $\delta \downarrow 0$, we get

$$
\begin{aligned}
&D^+\lambda_0(\kappa) \le (M-1)2d \qquad \forall M > 1\colon\ 2d\kappa\mathscr{I}(M) - \gamma \ge 0, \\
&D^-\lambda_0(\kappa) \ge -2d.
\end{aligned} \tag{53}
$$

(The condition in the first line of (53) guarantees that the first term in the right-hand side of (49) is the maximum because $\lambda_0(\kappa) \geq 0$.) Since $\lim_{M\to\infty} \mathscr{I}(M) = \infty$, it follows from (53) that $D^+\lambda_0$ and $D^-\lambda_0$ are bounded outside any neighborhood of $\kappa = 0$.

2. It remains to explain how to remove the restriction $\xi \leq 1$. Without this restriction (46) is no longer true, but by the Cauchy-Schwarz inequality we have

$$II \leq III \times IV \tag{54}$$

with

$$III = \left\{ \mathrm{E}_0\left(\exp\left[2\gamma \int_0^t \xi(X^{\kappa_1}(s),s)\,ds\right] \delta_0(X^{\kappa_1}(t)) \right) \right\}^{1/2} \tag{55}$$

and

$$\begin{aligned} IV &= \Bigg\{ \mathrm{E}_0\Bigg(\exp\left[2J(X^{\kappa_1};t)\log(\kappa_2/\kappa_1) - 4d(\kappa_2-\kappa_1)t\right] \\ &\qquad\qquad \times 1\!\!1\{J(X^{\kappa_1};t) > M2d\kappa_2 t\} \Bigg) \Bigg\}^{1/2} \\ &= \exp\left[\left(d\kappa_1 - 2d\kappa_2 + d(\kappa_2^2/\kappa_1)\right)t\right] \\ &\quad \times \Bigg\{ \mathrm{E}_0\Bigg(\exp\left[J(X^{\kappa_1};t)\log\left(\frac{\kappa_2^2/\kappa_1}{\kappa_1}\right) - 2d\left(\kappa_2^2/\kappa_1 - \kappa_1\right)t\right] \\ &\qquad\qquad \times 1\!\!1\{J(X^{\kappa_1};t) > M2d\kappa_2 t\} \Bigg) \Bigg\}^{1/2} \\ &= \exp\left[\left(d\kappa_1 - 2d\kappa_2 + d(\kappa_2^2/\kappa_1)\right)t\right] \left\{ \mathrm{P}_0\left(J\left(X^{\kappa_2^2/\kappa_1};t\right) > M2d\kappa_2 t \right) \right\}^{1/2}, \end{aligned} \tag{56}$$

where in the last line we use Girsanov's formula in the reverse direction (without ξ). By (12–13) and condition (7), we have $III \leq e^{c_0 t}$ ξ-a.s. for $t \geq 0$ and some $c_0 < \infty$. Therefore, combining (54–56), we get

$$II \leq \exp\left[\left(c_0 + d\kappa_1 - 2d\kappa_2 + d(\kappa_2^2/\kappa_1)\right)t\right] \left\{ \mathrm{P}_0\left(J\left(X^{\kappa_2^2/\kappa_1};t\right) > M2d\kappa_2 t \right) \right\}^{1/2} \tag{57}$$

instead of (45). The rest of the proof goes along the same lines as in (47–53).

3. Since $\mathscr{I}(M) > 0$ for all $M > 1$, it follows from (53) that $\limsup_{\kappa\to\infty} D^+\lambda_0(\kappa) \leq 0$. To prove that $\liminf_{\kappa\to\infty} D^-\lambda_0(\kappa) \geq 0$, we argue as follows. From (43) with $\kappa_1 = \kappa - \delta$ and $\kappa_2 = \kappa$, we get

$$
\begin{aligned}
&\mathrm{E}_0\left(\exp\left[\gamma\int_0^t \xi(X^\kappa(s),s)\,ds\right]\delta_0(X^\kappa(t))\right)\\
&=\mathrm{E}_0\left(\exp\left[\gamma\int_0^t \xi(X^{\kappa-\delta}(s),s)\,ds\right]\delta_0(X^{\kappa-\delta}(t))\right.\\
&\qquad\qquad \left.\times\exp\left[J(X^{\kappa-\delta};t)\log\left(\tfrac{\kappa}{\kappa-\delta}\right)-2d\delta t\right]\right)\\
&\geq e^{-2d\delta t}\left[\mathrm{E}_0\left(\exp\left[p\gamma\int_0^t \xi(X^{\kappa-\delta}(s),s)\,ds\right]\delta_0(X^{\kappa-\delta}(t))\right)\right]^{1/p}\\
&\qquad\qquad \times\left[\mathrm{E}_0\left(\exp\left[qJ(X^{\kappa-\delta};t)\ \log\left(\tfrac{\kappa}{\kappa-\delta}\right)\right]\right)\right]^{1/q}\\
&=e^{-2d\delta t}\times I\times II,
\end{aligned}
\tag{58}
$$

where we use the reverse Hölder inequality with $(1/p)+(1/q)=1$ and $-\infty<q<0<p<1$. By direct computation, we have

$$
\mathrm{E}_0\left(\exp\left[qJ(X^{\kappa-\delta};t)\ \log\left(\tfrac{\kappa}{\kappa-\delta}\right)\right]\right)=\exp\left[-2d(\kappa-\delta)\left[1-\left(\tfrac{\kappa}{\kappa-\delta}\right)^q\right]t\right] \tag{59}
$$

and hence

$$
\frac{1}{\delta t}\log\left(e^{-2d\delta t}\times II\right)=-2d-\frac{2d}{\delta q}(\kappa-\delta)\left[1-\left(\tfrac{\kappa}{\kappa-\delta}\right)^q\right]. \tag{60}
$$

Moreover, with the help of the additional assumption that $\xi\leq 1$, we can estimate

$$
I\geq\exp\left[-\left(\tfrac{1-p}{p}\right)\gamma t\right]\left[\mathrm{E}_0\left(\exp\left[\gamma\int_0^t \xi(X^{\kappa-\delta}(s),s)\,ds\right]\delta_0(X^{\kappa-\delta}(t))\right)\right]^{1/p}. \tag{61}
$$

Combining (58) and (60–61), we arrive at (insert $(1-p)/p=-1/q$)

$$
\begin{aligned}
&\frac{1}{\delta t}\left[\log\mathrm{E}_0\left(\exp\left[\gamma\int_0^t \xi(X^\kappa(s),s)\,ds\right]\delta_0(X^\kappa(t))\right)\right.\\
&\qquad\qquad\left.-\log\mathrm{E}_0\left(\exp\left[\gamma\int_0^t \xi(X^{\kappa-\delta}(s),s)\,ds\right]\delta_0(X^{\kappa-\delta}(t))\right)\right]\\
&\geq -2d-\frac{2d}{\delta q}(\kappa-\delta)\left[1-\left(\tfrac{\kappa}{\kappa-\delta}\right)^q\right]+\frac{\gamma}{\delta q}\\
&\qquad -\frac{1}{\delta q t}\log\mathrm{E}_0\left(\exp\left[\gamma\int_0^t \xi(X^{\kappa-\delta}(s),s)\,ds\right]\delta_0(X^{\kappa-\delta}(t))\right).
\end{aligned}
\tag{62}
$$

Let $t\to\infty$ to obtain

$$
\frac{1}{\delta}[\lambda_0(\kappa)-\lambda_0(\kappa-\delta)]\geq -2d-\frac{2d}{\delta q}(\kappa-\delta)\left[1-\left(\tfrac{\kappa}{\kappa-\delta}\right)^q\right]+\frac{1}{\delta q}[\gamma-\lambda_0(\kappa-\delta)]. \tag{63}
$$

Pick $q = -C/\delta$ with $C \in (0,\infty)$ and let $\delta \downarrow 0$, to obtain

$$D^- \lambda_0(\kappa) \geq -2d + \frac{2d\kappa}{C}\left(1 - e^{-C/\kappa}\right) - \frac{1}{C}\left[\gamma - \lambda_0(\kappa)\right]. \tag{64}$$

Let $\kappa \to \infty$ and use that $\lambda_0(\kappa) \geq 0$, to obtain

$$\liminf_{\kappa\to\infty} D^- \lambda_0(\kappa) \geq -\frac{\gamma}{C}. \tag{65}$$

Finally, let $C \to \infty$ to arrive at the claim. □

2.4 Proof of Theorem 1.2(iii)

Proof. Since ξ is assumed to be bounded from below, we may take $\xi \geq -1$ w.l.o.g., because we can scale γ. The proof of Theorem 1.2(iii) is based on the following lemma providing a lower bound for $\lambda_0(\kappa)$ when κ is small enough. Recall (20), and abbreviate

$$E_1(T) = \mathbb{E}\left(|I^\xi(0,T) - I^\xi(e,T)|\right), \qquad T > 0. \tag{66}$$

Lemma 2.1. *For $T \geq 1$ and $\kappa \downarrow 0$,*

$$\lambda_0(\kappa) \geq -\gamma\tfrac{1}{T} - 2d\kappa\tfrac{T-1}{T} + [1 + o_\kappa(1)]\,\tfrac{1}{T}\left[\tfrac{\gamma}{2}E_1(T-1) - \log(1/\kappa)\right]. \tag{67}$$

Proof. The proof comes in 2 steps. Recall (4) and (12–13), and write

$$\begin{aligned}&\lambda_0(\kappa)\\ &= \lim_{n\to\infty} \frac{1}{nT+1} \log \mathrm{E}_0\left(\exp\left[\gamma \int_0^{nT+1} \xi(X^\kappa(s), s)\,ds\right] \delta_0\left(X^\kappa(nT+1)\right)\right).\end{aligned} \tag{68}$$

1. Partition the time interval $[0, nT+1)$ as $[\cup_{j=1}^{n+1}\mathscr{B}_j] \cup [\cup_{j=1}^{n}\mathscr{C}_j]$ with

$$\begin{aligned}\mathscr{B}_j &= \left[(j-1)T, (j-1)T+1\right),\ 1 \leq j \leq n+1,\\ \mathscr{C}_j &= \left[(j-1)T+1, jT\right), \qquad 1 \leq j \leq n.\end{aligned} \tag{69}$$

Let

$$I_j^\xi(x) = \int_{\mathscr{C}_j} \xi(x,s)\,ds \tag{70}$$

and

$$Z_j^\xi = \mathrm{argmax}_{x\in\{0,e\}}\, I_j^\xi(x), \tag{71}$$

and define the event

$$A^\xi = \left[\bigcap_{j=1}^{n}\left\{X^\kappa(t) = Z_j^\xi\ \forall t \in \mathscr{C}_j\right\}\right] \cap \{X^\kappa(nT+1) = 0\}. \tag{72}$$

We may estimate

$$\begin{aligned}
&\mathrm{E}_0\left(\exp\left[\gamma\int_0^{nT+1}\xi(X^\kappa(s),s)\,ds\right]\delta_0\big(X^\kappa(nT+1)\big)\right)\\
&\qquad\geq \mathrm{E}_0\left(\exp\left[\gamma\int_0^{nT+1}\xi(X^\kappa(s),s)\,ds\right]1\!\!1_{A^\xi}\right)\\
&\qquad\geq e^{-\gamma(n+1)}\exp\left(\gamma\sum_{j=1}^n \max\left\{I_j^\xi(0),I_j^\xi(e)\right\}\right)\mathrm{P}_0\big(A^\xi\big).
\end{aligned}\tag{73}$$

By the ergodic theorem (recall condition (6)), we have

$$\begin{aligned}
&\sum_{j=1}^n \max\left\{I_j^\xi(0),I_j^\xi(e)\right\}\\
&= [1+o_n(1)]\,n\,\mathbb{E}\big(\max\left\{I_1^\xi(0),I_1^\xi(e)\right\}\big) \quad \xi\text{-a.s. as } n\to\infty.
\end{aligned}\tag{74}$$

Moreover, we have

$$\mathrm{P}_0\big(A^\xi\big)\geq\Big(\min\left\{p_1^\kappa(0),p_1^\kappa(e)\right\}\Big)^{n+1}e^{-2d\kappa n(T-1)}=\big(p_1^\kappa(e)\big)^{n+1}e^{-2d\kappa n(T-1)},\tag{75}$$

where in the right-hand side the first term is a lower bound for the probability that X^κ moves from 0 to e or vice-versa in time 1 in each of the time intervals $\mathscr{B}_j$, while the second term is the probability that X^κ makes no jumps in each of the time intervals $\mathscr{C}_j$.

2. Combining (68) and (73–75), and using that $p_1^\kappa(e)=\kappa[1+o_\kappa(1)]$ as $\kappa\downarrow 0$, we obtain that

$$\begin{aligned}
\lambda_0(\kappa)&\geq -\gamma\tfrac{1}{T}-2d\kappa\tfrac{T-1}{T}\\
&\qquad+[1+o_\kappa(1)]\,\frac{1}{T}\Big[\gamma\mathbb{E}\big(\max\left\{I_1^\xi(0),I_1^\xi(e)\right\}\big)-\log(1/\kappa)\Big].
\end{aligned}\tag{76}$$

Because $I_1^\xi(0)$ and $I_1^\xi(e)$ have zero mean, we have

$$\mathbb{E}\big(\max\left\{I_1^\xi(0),I_1^\xi(e)\right\}\big)=\tfrac{1}{2}\,\mathbb{E}\big(\big|I_1^\xi(0)-I_1^\xi(e)\big|\big).\tag{77}$$

The expectation in the right-hand side equals $E_1(T-1)$ because $|\mathscr{C}_1|=T-1$ (recall (66)), and so we get the claim. □

Using Lemma 2.1, we can now complete the proof of Theorem 1.2(iii). By condition (21), for every $c\in(0,\infty)$ we have $E_1(T)\geq c\log T$ for T large enough (depending on c). Pick $\chi\in(0,1)$ and $T=T(\kappa)=\kappa^{-\chi}$ in (67) and let $\kappa\downarrow 0$. Then we obtain

$$\lambda_0(\kappa)\geq[1+o_\kappa(1)]\left\{-\gamma\kappa^\chi-2d\kappa+\left[\tfrac{1}{2}c\gamma\chi-1\right]\kappa^\chi\log(1/\kappa)\right\}.\tag{78}$$

Finally, pick c large enough so that $\frac{1}{2}c\gamma\chi > 1$. Then, because $\lambda_0(0) = 0$, (78) implies that, for $\kappa \downarrow 0$,

$$\lambda_0(\kappa) - \lambda_0(0) \geq [1 + o_\kappa(1)]\,[\tfrac{1}{2}c\gamma\chi - 1]\,\kappa^\chi \log(1/\kappa), \tag{79}$$

which shows that $\kappa \mapsto \lambda_0(\kappa)$ is not Lipschitz at 0. □

2.5 Proof of Theorem 1.3(i)

Proof. Recall (20) and define

$$\begin{aligned} E_k(T) &= \mathbb{E}\big(|I^\xi(0,T) - I^\xi(e,T)|^k\big), \\ \bar{E}_k(T) &= \mathbb{E}\big(|I^\xi(0,T)|^k\big), \end{aligned} \qquad T > 0,\, k \in \mathbb{N}. \tag{80}$$

Estimate, for $N > 0$,

$$\begin{aligned} E_1(T) &= \mathbb{E}\big(|I^\xi(0,T) - I^\xi(e,T)|\big) \\ &\geq \tfrac{1}{2N}\,\mathbb{E}\left(|I^\xi(0,T) - I^\xi(e,T)|^2\, \mathbb{1}_{\{|I^\xi(0,T)|\leq N \text{ and } |I^\xi(e,T)|\leq N\}}\right) \\ &= \tfrac{1}{2N}\left[E_2(T) - \mathbb{E}\left(|I^\xi(0,T) - I^\xi(e,T)|^2\, \mathbb{1}_{\{|I^\xi(0,T)|>N \text{ or } |I^\xi(e,T)|>N\}}\right)\right]. \end{aligned} \tag{81}$$

By Cauchy-Schwarz,

$$\begin{aligned} &\mathbb{E}\left(|I^\xi(0,T) - I^\xi(e,T)|^2\, \mathbb{1}_{\{|I^\xi(0,T)|>N \text{ or } |I^\xi(e,T)|>N\}}\right) \\ &\qquad \leq [E_4(T)]^{1/2}\left[\mathbb{P}\left(|I^\xi(0,T)| > N \text{ or } |I^\xi(e,T)| > N\right)\right]^{1/2}. \end{aligned} \tag{82}$$

Moreover, by condition (6), $E_4(T) \leq 16\bar{E}_4(T)$ and

$$\mathbb{P}\left(|I^\xi(0,T)| > N \text{ or } |I^\xi(e,T)| > N\right) \leq \tfrac{2}{N^2}\,\bar{E}_2(T) \leq \tfrac{2}{N^2}\,[\bar{E}_4(T)]^{1/2}. \tag{83}$$

By condition (22), there exist an $a > 0$ such that $E_2(T) \geq aT$ and a $b < \infty$ such that $\bar{E}_4(T) \leq bT^2$ for T large enough. Therefore, combining (81–83) and picking $N = cT^{1/2}$ with $c > 0$, we obtain

$$E_1(T) \geq A\,T^{1/2} \text{ with } A = \tfrac{1}{2c}\left(a - 2^{5/2}b^{3/4}\tfrac{1}{c}\right), \tag{84}$$

where we note that $A > 0$ for c large enough. Inserting this bound into Lemma 2.1 and picking $T = T(\kappa) = B[\log(1/\kappa)]^2$ with $B > 0$, we find that, for $\kappa \downarrow 0$,

$$\lambda_0(\kappa) \geq C\,[\log(1/\kappa)]^{-1}\,[1 + o_\kappa(1)] \text{ with } C = \tfrac{1}{B}\left(\tfrac{1}{2}\gamma AB^{1/2} - 1\right). \tag{85}$$

Since $C > 0$ for $A > 0$ and B large enough, this proves the claim in (25). □

2.6 Proof of Theorem 1.3(iii)

The proof borrows from Carmona and Molchanov [6], Section IV.3.

Proof. Recall (4) and (12–13), estimate

$$\lambda_0(\kappa) \le \limsup_{n\to\infty} \frac{1}{nT} \log \mathrm{E}_0 \left(\exp\left[\gamma \int_0^{nT} \xi(X^\kappa(s),s)\,ds \right] \right), \tag{86}$$

and pick

$$T = T(\kappa) = K \log(1/\kappa), \qquad K \in (0,\infty), \tag{87}$$

where K is to be chosen later. Partition the time interval $[0,nT)$ into n disjoint time intervals $\mathscr{I}_j$, $1 \le j \le n$, defined in (32). Let N_j, $1 \le j \le n$, be the number of jumps of X^κ in the time interval $\mathscr{I}_j$, and call $\mathscr{I}_j$ *black* when $N_j > 0$ and *white* when $N_j = 0$. Using Cauchy-Schwarz, we can split $\lambda_0(\kappa)$ into a black part and a white part, and estimate

$$\lambda_0(\kappa) \le \mu_0^{(\mathrm{b})}(\kappa) + \mu_0^{(\mathrm{w})}(\kappa), \tag{88}$$

where

$$\mu_0^{(\mathrm{b})}(\kappa) = \limsup_{n\to\infty} \frac{1}{2nT} \log \mathrm{E}_0 \left(\exp\left[2\gamma \sum_{\substack{j=1\\ N_j>0}}^{n} \int_{\mathscr{I}_j} \xi(X^\kappa(s),s)\,ds \right] \right), \tag{89}$$

$$\mu_0^{(\mathrm{w})}(\kappa) = \limsup_{n\to\infty} \frac{1}{2nT} \log \mathrm{E}_0 \left(\exp\left[2\gamma \sum_{\substack{j=1\\ N_j=0}}^{n} \int_{\mathscr{I}_j} \xi(X^\kappa(s),s)\,ds \right] \right). \tag{90}$$

Lemma 2.2. *If ξ is bounded from above, then there exists a $\delta > 0$ such that*

$$\limsup_{\kappa\downarrow 0} (1/\kappa)^\delta \mu_0^{(\mathrm{b})}(\kappa) \le 0. \tag{91}$$

Lemma 2.3. *If ξ satisfies condition* (24), *then*

$$\limsup_{\kappa\downarrow 0} \frac{\log(1/\kappa)}{\log\log(1/\kappa)} \mu_0^{(\mathrm{w})}(\kappa) < \infty. \tag{92}$$

Theorem 1.3(ii) follows from (88) and Lemmas 2.2–2.3. □

We first give the proof of Lemma 2.2.

Proof. Let $N^{(\mathrm{b})} = |\{1 \le j \le n\colon N_j > 0\}|$ be the number of black time intervals. Since ξ is bounded from above (w.l.o.g. we may take $\xi \le 1$, because we can scale γ), we have

$$
\begin{aligned}
&\frac{1}{2nT}\log \mathrm{E}_0\left(\exp\left[2\gamma \sum_{\substack{j=1\\ N_j>0}}^{n} \int_{\mathscr{I}_j} \xi(X^\kappa(s),s)\,ds\right]\right)\\
&\qquad \le \frac{1}{2nT}\log \mathrm{E}_0\Big(\exp\big[2\gamma T N^{(\mathrm{b})}\big]\Big)\\
&\qquad = \frac{1}{2T}\log\left[\left(1-e^{-2d\kappa T}\right)e^{2\gamma T}+e^{-2d\kappa T}\right]\\
&\qquad \le \frac{1}{2T}\log\left[2d\kappa T e^{2\gamma T}+1\right]\\
&\qquad \le \frac{1}{2T}\,2d\kappa T e^{2\gamma T}\\
&\qquad = d\kappa^{1-2\gamma K},
\end{aligned}
\tag{93}
$$

where the first equality uses that the distribution of $N^{(\mathrm{b})}$ is $\mathrm{BIN}(n, 1-e^{-2d\kappa T})$, and the second equality uses (87). It follows from (89) and (93) that $\mu_0^{(\mathrm{b})}(\kappa) \le d\kappa^{1-2\gamma K}$. The claim in (91) therefore follows by picking $0 < K < 1/2\gamma$ and letting $\kappa \downarrow 0$. □

We next give the proof of Lemma 2.3.

Proof. The proof comes in 5 steps.

1. We begin with some definitions. Define $\Gamma = (\Gamma_1,\dots,\Gamma_n)$ with

$$
\Gamma_j = \begin{cases} \{\Delta_1,\dots,\Delta_{N_j}\} & \text{if } \mathscr{I}_j \text{ is black},\\ \emptyset & \text{if } \mathscr{I}_j \text{ is white}, \end{cases} \tag{94}
$$

where $\Delta_1,\dots,\Delta_{N_j}$ are the jumps of X^κ in the time interval $\mathscr{I}_j$ (which take values in the unit vectors of $\mathbb{Z}^d$). Next, let

$$
\Psi = \{\chi\colon\ \Gamma = \chi\}, \qquad \chi = (\chi_1,\dots,\chi_n), \tag{95}
$$

denote the set of possible outcomes of Γ. Since X^κ is stepping at rate $2d\kappa$, the random variable Γ has distribution

$$
\mathrm{P}_0(\Gamma=\chi) = e^{-2d\kappa n T}\prod_{j=1}^{n}\frac{(2d\kappa T)^{n_j(\chi)}}{n_j(\chi)!}, \qquad \chi\in\Psi, \tag{96}
$$

with $n_j(\chi) = |\chi_j| = |\{\chi_{j,1},\dots,\chi_{j,n_j(\chi)}\}|$ the number of jumps in χ_j. For $\chi\in\Psi$, we define the event

$$
A^{(n)}(\chi;\lambda) = \left\{ \sum_{\substack{j=1\\ n_j(\chi)=0}}^{n} \int_{\mathscr{I}_j} \xi(x_j(\chi),s)\,ds \ge \lambda \right\}, \qquad \chi\in\Psi,\ \lambda>0, \tag{97}
$$

where $x_j(\chi) = \sum_{i=1}^{j-1}\sum_{k=1}^{n_i(\chi)}\chi_{i,k}$ is the location of χ at the start of χ_j, and λ is to be chosen later. We further define

$$k_l(\chi) = |\{1 \le j \le n\colon\ n_j(\chi) = l\}|, \qquad l \ge 0, \tag{98}$$

which counts the time intervals in which χ makes l jumps, and we note that

$$\sum_{l=0}^{\infty} k_l(\chi) = n. \tag{99}$$

2. With the above definitions, we can now start our proof. Fix $\chi \in \Psi$. By (97) and the exponential Chebychev inequality, we have

$$\begin{aligned}
\mathbb{P}\big(A^{(n)}(\chi;\lambda)\big) &= \mathbb{P}\left(\sum_{\substack{j=1\\ n_j(\chi)=0}}^{n} \int_{\mathscr{I}_j} \xi(x_j(\chi),s)\,ds \ge \lambda\right)\\
&\le \inf_{\mu>0} e^{-\mu\lambda}\,\mathbb{E}\left(\prod_{\substack{j=1\\ n_j(\chi)=0}}^{n} \exp\left[\mu \int_{\mathscr{I}_j} \xi(x_j(\chi),s)\,ds\right]\right)\\
&\le \inf_{\mu>0} e^{-\mu\lambda}\left[\sup_{\eta\in\Omega} \mathbb{E}_\eta\left(e^{\mu I^{\xi}(0,T)}\right)\right]^{k_0(\chi)},
\end{aligned} \tag{100}$$

where in the second inequality we use the Markov property of ξ at the beginning of the white time intervals, and take the supremum over the starting configuration at the beginning of each of these intervals in order to remove the dependence on $\xi_{(j-1)T}$, $1 \le j \le n$ with $n_j(\chi) = 0$, after which we can use (6) and (20). Next, using condition (24) and choosing $\mu = b\lambda/k_0(\chi)T$, we obtain from (100) that

$$\mathbb{P}\big(A^{(n)}(\chi;\lambda)\big) \le \exp\left[-\frac{b\lambda^2}{k_0(\chi)T}\right]\left\{\exp\left[c\left(\frac{b\lambda}{k_0(\chi)T}\right)^2 T\right]\right\}^{k_0(\chi)}, \tag{101}$$

where c is the constant in condition (24). (Note that $A^{(n)}(\chi;\lambda) = \emptyset$ when $k_0(\chi) = 0$, which is a trivial case that can be taken care of afterwards.) By picking $b = 1/2c$, we obtain

$$\mathbb{P}\big(A^{(n)}(\chi;\lambda)\big) \le \exp\left[-\frac{1}{4c}\,\frac{\lambda^2}{k_0(\chi)T}\right]. \tag{102}$$

3. Our next step is to choose λ, namely,

$$\lambda = \lambda(\chi) = \sum_{l=0}^{\infty} a_l k_l(\chi) \tag{103}$$

with

$$\begin{aligned}
a_0 &= K' \log\log(1/\kappa), & K' &\in (0,\infty),\\
a_l &= lK \log(1/\kappa), & l &\ge 1,
\end{aligned} \tag{104}$$

where K is the constant in (87). It follows from (102) after substitution of (103–104) that (recall (87))

$$\mathbb{P}\Big(A^{(n)}(\chi;\lambda)\Big) \le \prod_{l=0}^{\infty} e^{u_l k_l(\chi)} \qquad \forall \chi \in \Psi, \tag{105}$$

where we abbreviate

$$u_0 = -\frac{1}{4cT}a_0^2, \qquad u_l = -\frac{1}{2cT}a_0 a_l = -\frac{1}{2c}a_0 l, \quad l \ge 1. \tag{106}$$

Summing over χ, we obtain

$$\begin{aligned}
\sum_{\chi\in\Psi} \mathbb{P}\Big(A^{(n)}(\chi;\lambda)\Big) &\le \sum_{\chi\in\Psi}\left(\prod_{l=0}^{\infty} e^{u_l k_l(\chi)}\right) \\
&= \sum_{\substack{(k_l)_{l=0}^{\infty} \\ \sum_{l=0}^{\infty} k_l = n}} \left(\frac{n!}{\prod_{l=0}^{\infty} k_l!}\right)\left(\prod_{l=0}^{\infty}(2d)^{lk_l}\right)\left(\prod_{l=0}^{\infty} e^{u_l k_l}\right) \\
&= \Big(\sum_{l=0}^{\infty}(2d)^l e^{u_l}\Big)^n,
\end{aligned} \tag{107}$$

where we use that for any sequence $(k_l)_{l=0}^{\infty}$ such that $\sum_{l=0}^{\infty} k_l = n$ (recall (99)) the number of $\chi \in \Psi$ such that $k_l(\chi) = k_l$, $l \ge 0$, equals $(n!/\prod_{l=0}^{\infty} k_l!)\prod_{l=0}^{\infty}(2d)^{lk_l}$ (note that there are $(2d)^l$ different χ_j with $|\chi_j| = l$ for each $1 \le j \le n$).

4. By (87) and (104), $T \to \infty$, $a_0 \to \infty$ and $a_0^2/T \downarrow 0$ as $\kappa \downarrow 0$. Hence, for $\kappa \downarrow 0$,

$$\begin{aligned}
\sum_{l=0}^{\infty}(2d)^l e^{u_l} &= \exp\left[-\frac{1}{4cT}a_0^2\right] + \frac{2d\exp\left[-\frac{1}{2c}a_0\right]}{1-2d\exp\left[-\frac{1}{2c}a_0\right]} \\
&= 1 - [1+o_\kappa(1)]\,\frac{[K'\log\log(1/\kappa)]^2}{8cK\log(1/\kappa)} + [1+o_\kappa(1)]\,2d[\log(1/\kappa)]^{-K'/2c} \\
&= 1 - [1+o_\kappa(1)]\,\frac{[K'\log\log(1/\kappa)]^2}{8cK\log(1/\kappa)} < 1,
\end{aligned} \tag{108}$$

where the last equality holds provided we pick $K' > 2c$. It follows from (107–108) that, for $\kappa \downarrow 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\bigcup_{\chi\in\Psi} A^{(n)}(\chi;\lambda)\right) < \infty. \tag{109}$$

Hence, recalling (97), we conclude that, by the Borel-Cantelli lemma, ξ-a.s. there exists an $n_0(\xi) \in \mathbb{N}$ such that, for all $n \ge n_0(\xi)$,

$$\sum_{\substack{j=1\\ n_j=0}}^{n} \int_{\mathscr{I}_j} \xi(x_j(\chi),s)\,ds \le \lambda = \sum_{l=0}^{\infty} a_l k_l(\chi) \qquad \forall \chi \in \Psi. \tag{110}$$

5. The estimate in (110) allows us to proceed as follows. Combining (96), (98) and (110), we obtain, for $n \ge n_0(\xi,\delta,\kappa)$,

$$\begin{aligned} &\mathrm{E}_0\left(\exp\left[2\gamma \sum_{\substack{j=1\\ N_j=0}}^{n} \int_{\mathscr{I}_j} \xi(X^\kappa(s),s)\,ds\right]\right)\\ &\qquad \le e^{-2d\kappa nT} \sum_{\chi\in\Psi} \left(\prod_{l=0}^{\infty}\left(\frac{(2d\kappa T)^l}{l!}\right)^{k_l(\chi)}\right)\left(\prod_{l=0}^{\infty} e^{2\gamma a_l k_l(\chi)}\right). \end{aligned} \tag{111}$$

Via the same type of computation as in (107), this leads to

$$\begin{aligned} &\mathrm{E}_0\left(\exp\left[2\gamma \sum_{\substack{j=1\\ N_j=0}}^{n} \int_{\mathscr{I}_j} \xi(X^\kappa(s),s)\,ds\right]\right)\\ &\quad \le e^{-2d\kappa nT} \sum_{\substack{(k_l)_{l=0}^\infty\\ \sum_{l=0}^\infty k_l = n}} \left(\frac{n!}{\prod_{l=0}^\infty k_l!}\right)\left(\prod_{l=0}^{\infty}(2d)^{lk_l}\right)\left(\prod_{l=0}^{\infty}\left(\frac{(2d\kappa T)^l}{l!}\,e^{2\gamma a_l}\right)^{k_l}\right)\\ &\quad = e^{-2d\kappa nT}\left(\sum_{l=0}^{\infty}\frac{((2d)^2\kappa T)^l}{l!}\,e^{2\gamma a_l}\right)^n. \end{aligned} \tag{112}$$

Hence

$$\begin{aligned} &\frac{1}{2nT}\log \mathrm{E}_0\left(\exp\left[2\gamma \sum_{\substack{j=1\\ N_j=0}}^{n} \int_{\mathscr{I}_j} \xi(X^\kappa(s),s)\,ds\right]\right)\\ &\qquad \le -d\kappa + \frac{1}{2T}\log\left(\sum_{l=0}^{\infty}\frac{((2d)^2\kappa T)^l}{l!}\,e^{2\gamma a_l}\right). \end{aligned} \tag{113}$$

Note that the r.h.s. of (113) does not depend on n. Therefore, letting $n\to\infty$ and recalling (90), we get

$$\mu_0^{(\mathrm{w})}(\kappa) \le -d\kappa + \frac{1}{2T}\log\left(\sum_{l=0}^{\infty}\frac{((2d)^2\kappa T)^l}{l!}\,e^{2\gamma a_l}\right). \tag{114}$$

Finally, by (87) and (104),

$$\begin{aligned} \sum_{l=0}^{\infty}\frac{((2d)^2\kappa T)^l}{l!}e^{2\gamma a_l} &= [\log(1/\kappa)]^{2\gamma K'} + \sum_{l=1}^{\infty}\frac{\left((2d)^2\kappa^{1-2\gamma K}K\log(1/\kappa)\right)^l}{l!}\\ &= [\log(1/\kappa)]^{2\gamma K'} + o_\kappa(1), \qquad \kappa\downarrow 0, \end{aligned} \tag{115}$$

where we recall from the proof of Lemma 2.2 that $0 < K < 1/2\gamma$. Hence

$$\mu_0^{(\mathrm{w})}(\kappa) \leq [1 + o_\kappa(1)]\, \frac{\gamma K' \log\log(1/\kappa)}{K \log(1/\kappa)}, \qquad \kappa \downarrow 0, \tag{116}$$

which proves the claim in (92). $\square$

2.7 Proof of Theorem 1.3(ii)

Theorem 1.3(ii) follows from (88), Lemma 2.2 and the following modification of Lemma 2.3.

Lemma 2.4. *If ξ satisfies condition (23) and is bounded from above, then*

$$\limsup_{\kappa \downarrow 0}\, [\log(1/\kappa)]^{1/6}\, \mu^{(\mathrm{w})}(\kappa) < \infty. \tag{117}$$

Proof. Most of Steps 1–5 in the proof of Lemma 2.3 can be retained.

1. Recall (94–99). Let

$$f_\varepsilon(T) = \sup_{\eta \in \Omega} \mathbb{P}_\eta\left(\int_0^T \xi(0,s)\, ds > \varepsilon T\right), \qquad T > 0, \varepsilon = T^{-1/6}. \tag{118}$$

Since $\xi \leq 1$ w.l.o.g., we may estimate

$$\sum_{\substack{j=1 \\ n_j(\chi)=0}}^{n} \int_{\mathscr{I}_j} \xi(x_j(\chi), s)\, ds \preceq ZT + (k_0(\chi) - Z)\varepsilon T, \tag{119}$$

where $\preceq$ means "stochastically dominated by", and Z is the random variable with distribution $\mathrm{P}^* = \mathrm{BIN}(k_0(\chi), f_\varepsilon(T))$. With the help of (119), the estimate in (100) can be replaced by

$$\begin{aligned} \mathbb{P}\big(A^{(n)}(\chi;\lambda)\big) &\leq \mathrm{P}^*\big(ZT + (k_0(\chi) - Z)\varepsilon T \geq \lambda\big) \\ &\leq \inf_{\mu > 0} e^{-\mu\lambda}\, \mathrm{E}^*\Big(e^{\mu[ZT + (k_0(\chi) - Z)\varepsilon T]}\Big) \\ &= \inf_{\mu > 0} e^{-\mu\lambda} \left\{ f_\varepsilon(T)\, e^{\mu T} + [1 - f_\varepsilon(T)]\, e^{\mu\varepsilon T} \right\}^{k_0(\chi)}. \end{aligned} \tag{120}$$

Using condition (24), which implies that there exists a $C \in (0,\infty)$ such that $f_\varepsilon(T) \leq e^{-C\varepsilon^2 T}$ for T large enough, and choosing $\mu = C\varepsilon^2\lambda/2k_0(\chi)T$, we obtain from (120) that, for T large enough,

$$\mathbb{P}\big(A^{(n)}(\chi;\lambda)\big) \leq \exp\left[-\frac{C\varepsilon^2\lambda^2}{2k_0(\chi)T}\right]\left\{\exp\left[\frac{C\varepsilon^2\lambda}{2k_0(\chi)}-C\varepsilon^2T\right]+\exp\left[\frac{C\varepsilon^3\lambda}{2k_0(\chi)}\right]\right\}^{k_0(\chi)}. \tag{121}$$

2. We choose λ as

$$\lambda=\lambda(\chi)=\sum_{l=0}^{\infty} b_l k_l(\chi) \tag{122}$$

with

$$\begin{aligned} b_0 &= 2\varepsilon K\log(1/\kappa)=2\varepsilon T,\\ b_l &= lK\log(1/\kappa)=lT, \quad l\geq 1. \end{aligned} \tag{123}$$

Note that this differs from (104) only for $l=0$, and that (99) implies, for T large enough,

$$\lambda \geq n2\varepsilon T. \tag{124}$$

3. Abbreviate the two exponentials between the braces in the right-hand side of (101) by I and II. Fix $A\in(1,2)$. In what follows we distinguish between two cases: $\lambda > Ak_0(\chi)T$ and $\lambda \leq Ak_0(\chi)T$.

$\underline{\lambda > Ak_0(\chi)T}$: Abbreviate $\alpha_1=\frac{1}{4}A>0$. Neglect the term $-C\varepsilon^2T$ in I, to estimate, for T large enough,

$$\begin{aligned} I+II &\leq \exp\left[\frac{C\varepsilon^2\lambda}{2k_0(\chi)}\right]\left(1+\exp\left[-\frac{C\varepsilon^2\lambda}{4k_0(\chi)}\right]\right)\\ &\leq \exp\left[\frac{C\varepsilon^2\lambda}{2k_0(\chi)}\right]\left(1+e^{-\alpha_1 C\varepsilon^2 T}\right). \end{aligned} \tag{125}$$

This yields

$$\mathbb{P}\big(A^{(n)}(\chi;\lambda)\big) \leq \exp\left[-\frac{C\varepsilon^2\lambda^2}{2k_0(\chi)T}+\frac{C\varepsilon^2\lambda}{2}\right]\exp\left[k_0(\chi)e^{-\alpha_1 C\varepsilon^2T}\right]. \tag{126}$$

$\underline{\lambda \leq Ak_0(\chi)T}$: Abbreviate $\alpha_2=1-\frac{1}{2}A>0$. Note that $I\leq \exp[-\alpha_2 C\varepsilon^2 T]$ and $II\geq 1$, to estimate

$$I+II \leq II\left(1+e^{-\alpha_2 C\varepsilon^2 T}\right). \tag{127}$$

This yields

$$\mathbb{P}\big(A^{(n)}(\chi;\lambda)\big) \leq \exp\left[-\frac{C\varepsilon^2\lambda^2}{2k_0(\chi)T}+\frac{C\varepsilon^3\lambda}{2}\right]\exp\left[k_0(\chi)e^{-\alpha_2 C\varepsilon^2T}\right]. \tag{128}$$

We can combine (126) and (128) into the single estimate

$$\mathbb{P}\big(A^{(n)}(\chi;\lambda)\big) \leq \exp\left[-\frac{C'\varepsilon^4\lambda^2}{2k_0(\chi)T}\right]\exp\left[k_0(\chi)e^{-\alpha CT}\right] \tag{129}$$

for some $C' = C' \in (0,\infty)$ with $\alpha = \alpha_1 \wedge \alpha_2 > 0$. To see why, put $x = \lambda / k_0(\chi)T$, and rewrite the exponent of the first exponential in the right-hand side of (126) and (128) as

$$\tfrac{1}{2} C\varepsilon^2 k_0(\chi) T\,(-x^2 + x), \quad \text{respectively,} \quad \tfrac{1}{2} C\varepsilon^2 k_0(\chi) T\,(-x^2 + \varepsilon x). \tag{130}$$

In the first case, since $A > 1$, there exists a $B > 0$ such that $-x^2 + x \leq -Bx^2$ for all $x \geq A$. In the second case, there exists a $B > 0$ such that $-x^2 + \varepsilon x \leq -B\varepsilon^2 x^2$ for all $x \geq 2\varepsilon$. But (124) ensures that $x \geq 2\varepsilon n / k_0(\chi) \geq 2\varepsilon$. Thus, we indeed get (129) with $C' = CB$.

4. The same estimates as in (105–107) lead us to

$$\sum_{\chi \in \Psi} \mathbb{P}\Big(A^{(n)}(\chi;\lambda)\Big) \leq \Big(\sum_{l=0}^{\infty} (2d)^l e^{v_l}\Big)^n. \tag{131}$$

with

$$v_0 = -\frac{C'\varepsilon^4}{2T} b_0^2 + e^{-\alpha C \varepsilon^2 T}, \qquad v_l = -\frac{C'\varepsilon^4}{T} b_0 b_l = -C'\varepsilon^4 b_0 l, \quad l \geq 1. \tag{132}$$

By (87) and (123), we have

$$\begin{aligned} \sum_{l=0}^{\infty} (2d)^l e^{v_l} &= \exp\left[-\frac{C'\varepsilon^4}{2T} b_0^2 + e^{-\alpha C \varepsilon^2 T}\right] + \frac{2d \exp\left[-C'\varepsilon^4 b_0\right]}{1 - 2d \exp[-C'\varepsilon^4 b_0]} \\ &= \exp\left[-2C' + e^{-\alpha C T^{2/3}}\right] + \frac{2d \exp[-2C' T^{1/6}]}{1 - 2d \exp[-2C' T^{1/6}]} \\ &< 1 \end{aligned} \tag{133}$$

for T large enough, i.e., κ small enough. This replaces (108). Therefore the analogues of (109–110) hold, i.e., ξ-a.s. there exists an $n_0(\xi) \in \mathbb{N}$ such that, for all $n \geq n_0(\xi)$,

$$\sum_{\substack{j=1 \\ n_j = 0}}^{n} \int_{\mathscr{I}_j} \xi(x_j(\chi), s)\, ds \leq \lambda = \sum_{l=0}^{\infty} b_l k_l(\chi) \qquad \forall \chi \in \Psi. \tag{134}$$

5. The same estimate as in (111–114) now lead us to

$$\mu_0^{(\mathrm{w})}(\kappa) \leq -d\kappa + \frac{1}{2T} \log\left(\sum_{l=0}^{\infty} \frac{((2d)^2 \kappa T)^l}{l!}\, e^{2\gamma b_l}\right). \tag{135}$$

Finally, by (87) and (123),

$$\sum_{l=0}^{\infty} \frac{((2d)^2 \kappa T)^l}{l!} e^{2\gamma b_l} = e^{4\gamma\varepsilon T} + \sum_{l=1}^{\infty} \frac{\big((2d)^2 \kappa^{1-2\gamma K} K \log(1/\kappa)\big)^l}{l!} \tag{136}$$
$$= e^{4\gamma\varepsilon T} + o_\kappa(1), \qquad \kappa \downarrow 0,$$

which replaces (115). Hence

$$\mu_0^{(\mathrm{w})}(\kappa) \leq [1 + o_\kappa(1)]\, 2\gamma\varepsilon, \qquad \kappa \downarrow 0, \tag{137}$$

which proves the claim in (117). □

3 Proof of Theorems 1.4–1.6

The proofs of Theorems 1.4–1.6 are given in Sections 3.1–3.3, respectively.

3.1 Proof of Theorem 1.4

Proof. For ISRW, SEP and SVM in the weakly catalytic regime, it is known that $\lim_{\kappa\to\infty} \lambda_1(\kappa) = \rho\gamma$ (recall Section 1.3.2). The claim therefore follows from the fact that $\rho\gamma \leq \lambda_0(\kappa) \leq \lambda_1(\kappa)$ for all $\kappa \in [0,\infty)$.
Note: The claim extends to non-symmetric voter models (see [17], Theorems 1.4–1.5). □

3.2 Proof of Theorem 1.5

Proof. It suffices to prove condition (22), because we saw in Section 2.5 that condition (22) implies (84), which is stronger than condition (21). Step 1 deals with $E_2(T)$, step 2 with $\bar{E}_4(T)$.

1. Let

$$C(x,t) = \mathbb{E}\big([\xi(0,0) - \rho][\xi(x,t) - \rho]\big), \qquad x \in \mathbb{Z}^d, t \geq 0, \tag{138}$$

denote the two-point correlation function of ξ. By condition (6), we have

$$E_2(T) = \int_0^T ds \int_0^T dt\ \mathbb{E}\big([\xi(0,s) - \xi(e,s)][\xi(0,t) - \xi(e,t)]\big)$$
$$= 4 \int_0^T ds \int_0^{T-s} dt\ [C(0,t) - C(e,t)]. \tag{139}$$

In what follows $G_d(x) = \int_0^\infty p_t(x) dt$, $x \in \mathbb{Z}^d$, denotes the Green function of simple random walk on $\mathbb{Z}^d$ stepping at rate 1 starting from 0, which is finite if and only if

$d \geq 3$. (Recall from Section 1.3.2 that SVM with a simple random walk transition kernel is of no interest in $d = 1, 2$.)

Lemma 3.1. *For $x \in \mathbb{Z}^d$ and $t \geq 0$,*

$$C(x,t) = \begin{cases} \rho\, p_t(x), & \text{ISRW},\\ \rho(1-\rho)\, p_t(x), & \text{SEP},\\ [\rho(1-\rho)/G_d(0)] \int_0^\infty p_{t+u}(x)\, du, & \text{SVM}. \end{cases} \tag{140}$$

Proof. For ISRW, we have

$$\xi(x,t) = \sum_{y \in \mathbb{Z}^d} \sum_{j=1}^{N^y} \delta_x\big(Y_j^y(t)\big), \qquad x \in \mathbb{Z}^d, t \geq 0, \tag{141}$$

where $\{N^y : y \in \mathbb{Z}^d\}$ are i.i.d. Poisson random variables with mean $\rho \in (0,\infty)$, and $\{Y_j^y : y \in \mathbb{Z}^d, 1 \leq j \leq N^y\}$ with $Y_j^y = (Y_j^y(t))_{t\geq 0}$ is a collection of independent simple random walks with jump rate 1 (Y_j^y is the j-th random walk starting from $y \in \mathbb{Z}^d$). Inserting (141) into (138), we get the first line in (140). For SEP and SVM, the claim follows via the graphical representation (see Gärtner, den Hollander and Maillard [14], Eq. (1.5.5), and [17], Lemma A.1, respectively). Recall from the remark made at the end of Section 1.1 that SVM requires the random walk transition kernel to be *transient*. □

For ISRW, (139–140) yield

$$\frac{1}{T} E_2(T) = 4\rho \int_0^T dt\, \frac{T-t}{T}\, [p_t(0) - p_t(e)], \tag{142}$$

where we note that $p_t(0) - p_t(e) \geq 0$ by the symmetry of the random walk transition kernel. Hence, by monotone convergence,

$$\lim_{T\to\infty} \frac{1}{T} E_2(T) = 4\rho \int_0^\infty dt\, [p_t(0) - p_t(e)], \tag{143}$$

which is a number in $(0,\infty)$ (see Spitzer [26], Sections 24 and 29). For SEP, the same computation applies with ρ replaced by $\rho(1-\rho)$. For SVM, (139–140) yield

$$\frac{1}{T} E_2(T) = 4\frac{\rho(1-\rho)}{G_d(0)} \int_0^\infty du\, \left[\frac{u(2T-u)}{2T}\, \mathbb{1}_{\{u\leq T\}} + \tfrac{1}{2} T\, \mathbb{1}_{\{u \geq T\}}\right] [p_u(0) - p_u(e)]. \tag{144}$$

Hence, by monotone convergence (estimate $\frac{1}{2}T \leq \frac{1}{2}u$ in the second term of the integrand),

$$\lim_{T\to\infty} \frac{1}{T} E_2(T) = 4\frac{\rho(1-\rho)}{G_d(0)} \int_0^\infty du\, u\, [p_u(0) - p_u(e)], \tag{145}$$

which again is a number in $(0,\infty)$ (see Spitzer [26], Section 24).

2. Let

$$C(x,t;y,u;z,v) = \mathbb{E}\big([\xi(0,0)-\rho][\xi(x,t)-\rho][\xi(y,u)-\rho][\xi(z,v)-\rho]\big),\\ x,y,z \in \mathbb{Z}^d,\ 0 \le t \le u \le v, \tag{146}$$

denote the four-point correlation function of ξ. Then, by condition (6),

$$\bar{E}_4(T) = 4! \int_0^T ds \int_0^{T-s} dt \int_t^{T-s} du \int_u^{T-s} dv\ C(0,t;0,u;0,v). \tag{147}$$

To prove the second part of (22), we must estimate $C(0,t;0,u;0,v)$. For ISRW, this can be done by using (141), for SEP by using the Markov property and the graphical representation. In both cases the computations are long but straightforward, with leading terms of the form

$$M p_a(0,0) p_b(0,0) \tag{148}$$

with a,b linear in t, u or v, and $M < \infty$. Each of these leading terms, after being integrated as in (147), can be bounded from above by a term of order T^2, and hence $\limsup_{T\to\infty} \bar{E}_4(T)/T^2 < \infty$. The details are left to the reader. □

Note: We expect the second part of condition (22) to hold also for SVM. However, the graphical representation, which is based on coalescing random walks, seems too cumbersome to carry through the computations.

3.3 Proof of Theorem 1.6

Proof. For ISRW in the strongly catalytic regime, we know that $\lambda_1(\kappa) = \infty$ for all $\kappa \in [0,\infty)$ (recall Fig. 3), while $\lambda_0(\kappa) < \infty$ for all $\kappa \in [0,\infty)$ (by Kesten and Sidoravicius [19], Theorem 2). □

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